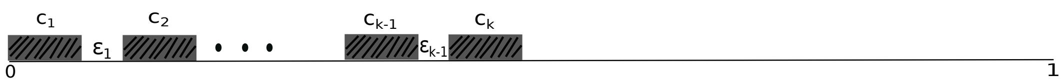
APPENDIX

Let $A = l_1 > c_1, \cdots, l_k > c_k$.



$$P(A \mid \vec{\epsilon}) = P(\text{Exactly } n-k \text{ cuts are made above } \sum c_i + \sum \epsilon_i$$
$$\& \text{ Exactly } k-1 \text{ cuts are made in the } \epsilon_i \text{ regions})$$

The volume of the simplex $\sum^{k-1} \epsilon_i < 1 - \sum^{k} c_i$ is

$$V = \int_0^{1-C} \int_0^{1-C-\epsilon_1} \cdots \int_0^{1-C-\epsilon_1-\cdots\epsilon_{k-2}} d\vec{\epsilon} = \frac{(1-C)^{k-1}}{(k-1)!}$$

So,

$$P(A) = \int P(A \mid \vec{\epsilon}) P(\vec{\epsilon}) d\vec{\epsilon} \quad \text{where} \quad P(\vec{\epsilon}) = \frac{(k-1)!}{(1-C)^{k-1}}$$

Lemma.

$$\int_0^{\beta} x(\beta - x)^k dx = \frac{\beta^{k+2}}{(k+2)(k+1)}$$

Proof.

$$\begin{aligned} \int_0^{\beta} x(\beta - x)^k dx &= -\frac{x(\beta - x)^{k+1}}{k+1}\Big|_0^{\beta} + \int_0^{\beta} \frac{(\beta - x)^{k+1}}{k+1} dx \\ &= 0 - \frac{(\beta - x)^{k+2}}{(k+2)(k+1)}\Big|_0^{\beta} \\ &= \frac{\beta^{k+2}}{(k+2)(k+1)} \end{aligned}$$

□

$$\int P(A \mid \vec{\epsilon})P(\vec{\epsilon})d\vec{\epsilon} = \binom{n-1}{n-k}\frac{(k-1)!}{(1-C)^{k-1}}\int_0^{1-C}\int_0^{1-C-\epsilon_1} \cdots \int_0^{1-C-\epsilon_1-\cdots\epsilon_{k-2}} \epsilon_1\cdots\epsilon_{k-1}(1-C-\epsilon_1-\cdots\epsilon_{k-1})^{n-k}d\epsilon_1\cdots d\epsilon_{k-1}$$

$$= \binom{n-1}{n-k}\frac{(k-1)!}{(1-C)^{k-1}}\int_0^{1-C}\int_0^{1-C-\epsilon_1} \cdots \int_0^{1-C-\epsilon_1-\cdots\epsilon_{k-3}} \frac{\epsilon_1\cdots\epsilon_{k-2}(1-C-\epsilon_1-\cdots\epsilon_{k-2})^{n-k+2}}{(n-k+2)(n-k+1)}d\epsilon_1\cdots d\epsilon_{k-2}$$

$$\vdots$$

$$= \binom{n-1}{n-k}\frac{(k-1)!}{(1-C)^{k-1}}\frac{(1-C)^{n-k+2k-2}}{(n-k+1)(n-k+2)\cdots(n-k+2k-2)(n-k+2k-3)}$$

$$= \binom{n-1}{n-k}\frac{(k-1)!}{(1-C)^{k-1}}\frac{(1-C)^{n+k-2}}{(n-k+1)(n-k+2)\cdots(n+k-2)(n+k-3)}$$

$$= \frac{(n-1)!}{(n-k)!}\frac{(1-C)^{n-1}}{(n-k+1)(n-k+2)\cdots(n+k-2)(n+k-3)}$$

$$= \frac{(n-1)!}{(n+k-2)!}(1-C)^{n-1}$$

We've got a problem here, the coefficient on the last line $\frac{(n-1)!}{(n+k-2)!} \neq 1$. No, worries. We can still save the argument by recognizing that some arrangements were unaccounted. The reciprocal $a_{n,k} = \frac{(n+k-2)!}{(n-1)!}$ is really

$$\frac{(n+k-2)!}{(n-1)!} = (\underbrace{n-1}_{n-1 \text{ cuts}} + \underbrace{k-1}_{\epsilon\text{-gaps}})! \Big/ (n-1)!$$

So, $a_{n,k}$ represents the number of *words* made with cuts α and $\{\epsilon_1, \ldots, \epsilon_k\}$. Possibilities for $a_{3,2}$ are

$$\alpha\epsilon\alpha, \quad \epsilon\alpha\alpha, \quad \alpha\alpha\epsilon.$$

Each *word* represents an specific order of cuts and $\epsilon-$gaps.